$SL_2(\mathbb{C})$-CHARACTER VARIETY OF A HYPERBOLIC LINK AND REGULATOR

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Abstract. In this paper, we study the $SL_2(\mathbb{C})$ character variety of a hyperbolic link in S^3. We analyze a special smooth projective variety Y^h arising from some 1-dimensional irreducible slices on the character variety. We prove that a natural symbol obtained from these 1-dimensional slices is a torsion in $K_2(\mathbb{C}(Y^h))$. By using the regulator map from K_2 to the corresponding Deligne cohomology, we get some variation formulae on some Zariski open subset of Y^h. From this we give some discussions on a possible parametrized volume conjecture for both hyperbolic links and knots.

1. Introduction

This is the sequel of our previous work [LW2] on the generalized volume conjecture for a hyperbolic knot in S^3. In this paper we shall study a hyperbolic link in S^3, and extend several results from the knot case. The main idea is to apply the regulator map in the K-theory to the $SL_2(\mathbb{C})$ character varieties of hyperbolic links.

For a link L in S^3, Kashaev ([Ka1]) introduced a sequence of complex numbers $\{K_N | N > 1, \text{odd integer}\}$, which were derived from a matrix version of the quantum dilogarithms. Kashaev's Volume Conjecture [Ka2] predicts that for any hyperbolic link L in S^3 the asymptotic behavior of his invariants $\{K_N\}$ regains the hyperbolic volume of $S^3 - L$. This was verified for the figure eight knot ([Ka2]). The Volume Conjecture provides an intriguing relationship between the quantum invariants and the hyperbolic volume, but we still do not fully understand it.

For the knot case, Murakami-Murakami ([MM]) showed the Kashaev invariants $\{K_N\}$ can be identified with the values of normalized colored Jones polynomial at the primitive N-th root of unity. From this, a new formulation of the Volume Conjecture is stated as that the asymptotic behavior of the colored Jones invariants of any knot equals the Gromov simplicial volume of its complement in S^3. This new version of the volume conjecture bridges the quantum invariants of the knot with its classical geometry and topology. However, this formulation does not fit well for links, since it does not hold for many split links (see [MMOTY]). Hence it is a very interesting question for us to see what is really behind the volume conjecture for links.

Following Witten's $SU(2)$ topological quantum field theory, Gukov [Guk] proposed a complex version of Chern-Simons theory and generalized the volume conjecture to a $\mathbb{C}^*$-parametrized version with parameter lying on the zero locus of the A-polynomial of the knot. In [LW2], we constructed a natural torsion element in the K_2 of the function field of the curve defined by the A-polynomial. We then showed that the part from the A-polynomial

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in Gukov's generalized volume conjecture can be interpreted by the regulator map on this torsion element. In particular, this implied the quantization condition posed by Gukov [Guk, Page 597].

It is natural to ask if there exists a parametrized volume conjecture for links in S^3 as Gukov did for the knot case. This is the motivation of this paper. Now we have to deal with two problems for links with more than one component. First, its $SL_2(\mathbb{C})$ character variety has dimension > 1. Hence it is not clear how to define an A-polynomial for such a link, which will contain the geometric information like volume and Chern-Simons as the knot case. Secondly, it is not clear how to relate the colored Jones polynomials to $SL_2(\mathbb{C})$ character variety of dimension > 1. In this paper, we shall focus on the first problem. We introduce n curves on the geometric component of the character variety. From these curves, we obtain an n-dimensional smooth projective variety Y^h, where n is the number of the components of the link. We construct a natural torsion element in K_2 of the function field of Y^h. By applying the regulator map on this torsion element, we get the variation formulae (Theorem 3.12) on some Zariski open subset of Y^h. When the link has one component, it recovers the results for hyperbolic knots. This suggests that there should exist a parametrized volume conjecture for hyperbolic links and the Y^h may provide a replacement of the locus of the A-polynomial of a knot. For the second problem, we give some speculations in the end of Section 4.

On the other hand, using dilogarithm, Dupont ([Dup]) constructed explicitly the Cheeger-Chern-Simons class associated to the second Chern polynomial. Apply it to a closed hyperbolic 3-manifold M, we get a number in $\mathbb{C}/\mathbb{Z}$. He (loc.cit.) showed that its imaginary part equals the hyperbolic volume of M and the real part is the Chern-Simons invariant of M. In general, for an odd dimension hyperbolic manifold of finite volume, Goncharov ([Gon]) constructed an element in the Quillen's algebraic K-group of $\mathbb{C}$ and proved that after applying it to the Borel regulator, we get the volume of the manifold. Our approach can be regarded as a family version of theirs for the $SL_2(\mathbb{C})$ character variety of a hyperbolic link.

Our paper is organized as follows. In section 2, we review the basics of the $SL_2(\mathbb{C})$ character variety of a hyperbolic link. We then study the properties of a smooth projective variety Y^h coming from the 1 dimensional slices of the character variety. In section 3, we recall the definitions and basic properties of K_2 of a commutative ring. Then we state and prove our main results in this section. In Section 4, we give some discussions related to a possible parametrized volume conjecture for hyperbolic links.

2. Character Variety of a hyperbolic link

2.1. Let L be a hyperbolic link in S^3 with n components $K_1, \ldots, K_n$. This means that the complement $S^3 - L$ carries a complete hyperbolic structure of finite volume. Let $N(L)$ be an open tubular neighborhood of L in S^3. Put $M_L = S^3 - N(L)$, then it is a compact 3-manifold with boundary ∂M_L a disjoint union of n tori $T_1, \ldots, T_n$. Note that $\pi_1(S^3 - L)$ and $\pi_1(M_L)$ are isomorphic. In the following, we shall identify them.

Let $R(M_L) = \mathrm{Hom}(\pi_1(M_L), SL_2(\mathbb{C}))$ and $R(T_i) = \mathrm{Hom}(\pi_1(T_i), SL_2(\mathbb{C}))$, $i = 1, \cdots, n$. $SL_2(\mathbb{C})$ acts on them by conjugation. According to [CS1], they are affine algebraic sets and so are the corresponding character varieties $X(M_L)$ and $X(T_i)$, which are the algebro-geometric quotients of $R(M_L)$ and $R(T_i)$ by $SL_2(\mathbb{C})$. We then have the canonical surjective morphisms $t : R(M_L) \longrightarrow X(M_L)$ and $t_i : R(T_i) \longrightarrow X(T_i)$ which map a representation to its character. Induced by the inclusions of $\pi_1(T_i)$ into $\pi_1(M_L)$, we have the restriction map:

$$r : X(M_L) \to X(T_1) \times \cdots \times X(T_n).$$

For the details on the character varieties, we refer to [CS1, Sha].

2.2. Let $\rho_0 : \pi_1(M_L) \to SL_2(\mathbb{C})$ be a representation associated to the complete hyperbolic structure on $S^3 - L$. Then it is irreducible. Denote χ_0 its character. Fix an irreducible component R_0 of $R(M_L)$ containing ρ_0. Let $X_0 = t(R_0)$, then X_0 is an affine variety of dimension n ([CS1, Sha]). We call it the geometric component of the character variety. Define Y_0 to be the Zariski closure of the image $r(X_0)$ in $X(T_1) \times \cdots \times X(T_n)$.

For $g \in \pi_1(M_L)$, there is a regular function $I_g : X_0 \to \mathbb{C}$ defined by $I_g(\chi) = \chi(g)$, for $\forall \chi \in X_0$. The following proposition was proved in [CS2].

Proposition 2.1. *Let γ_i be a non-contractible simple closed curve in the boundary torus T_i, $1 \le i \le n$. Let $g_i \in \pi_1(M_L)$ be an element whose conjugacy class corresponds to the free homotopy class of γ_i. Let k be an integer with $0 \le k \le n$, and let V be the algebraic subset of X_0 defined by the equations $I^2_{g_i}(\chi) = 4$, $k < i \le n$. Let V_0 denote an irreducible compoment of V containing χ_{ρ_0}. Then if χ is a point of V_0, i is an integer with $k < i \le n$, and g is an element of the subgroup (defined up to conjugacy) $Im(\pi_1(T_i) \to \pi_1(M_L))$, we have $I_g(\chi) = \pm 2$. Furthermore, if $k = 0$, then $V_0 = \{\chi_{\rho_0}\}$.*

Proof. See [CS2, Proposition 2, Page 539]. □

The following is a generalization of the knot case (c.f. [CS1, CS2]).

Proposition 2.2. *Y_0 is an n-dimensional affine variety.*

Proof. It is clear that Y_0 is an affine variety. We need to show that $\dim Y_0 = n$. Since $\dim X_0 = n$, $\dim Y_0 \le n$. Assume that $\dim Y_0 = m < n$. Then for $y \in r(X_0)$, every component of the fibre $r^{-1}(y)$ has dimension $\ge n - m \ge 1$. Take $y = r(\chi_0)$, then there is an irreducible component C of the fibre $r^{-1}(y)$ containing χ_0 and $\dim C \ge 1$. For each boundary torus T_i and a non-trivial $g_i \in \mathrm{Im}(\pi_1(T_i) \to \pi_1(M_L))$, consider the regular function $I_{g_i} : X_0 \longrightarrow \mathbb{C}$. For $\forall \chi \in C$, $I_{g_i}(\chi) = I_{g_i}(\chi_0)$. Since χ_0 is the character of the complete hyperbolic structure on M_L, $I^2_{g_i}(\chi) - 4 = I^2_{g_i}(\chi_0) - 4 = 0$ for $\forall \chi \in C$, $g_i \in \mathrm{Im}(\pi_1(T_i) \to \pi_1(M_L))$, $1 \le i \le n$. Now we fix n non-trivial $g_i \in \mathrm{Im}(\pi_1(T_i) \to \pi_1(M_L))$, $1 \le i \le n$. Consider the algebraic subset V of X_0 defined by the equations $I^2_{g_i} - 4 = 0$, $1 \le i \le n$. By its construction, C is contained in an irreducible component say,V_0 of V containing χ_0. Hence $\dim V_0 \ge 1$. On the other hand, by Proposition 2.1, $V_0 = \{\chi_0\}$, a contradiction. Therefore, $\dim Y_0 = n$. □

For every boundary torus T_i, fix a meridian-longitude basis $\{\mu_i, \lambda_i\}$ for $\pi_1(T_i) = H_1(T_i, \mathbb{Z})$. Given $1 \le i \le n$, we define X^i_0 as the subvariety of X_0 defined by the equations $I^2_{\mu_j} - 4 = 0$, $j \ne i$, $1 \le j \le n$. Let V_i be an irreducible component of X^i_0 containing χ_0.

Proposition 2.3. *For each $i = 1, \cdots, n$, V_i has dimension* 1.

Proof. Since X^i_0 is defined by $n - 1$ equations and $\dim X_0 = n$, every component of X^i_0 has dimension at least 1. Now assume that $\dim V_i \ge 2$. Let U be the subvariety of V_i defined by the equation $I^2_{\mu_i} - 4 = 0$ and let U_0 be the irreducible component of U containing χ_0. Then $\dim V_i \ge 2$ implies that $\dim U_0 \ge 1$. But this contradicts the last assertion in Proposition 2.1. Hence, $\dim V_i = 1$. □

Lemma 2.4. *Given a non-trivial $g_i \in Im(\pi_1(T_i) \to \pi_1(M_L))$, $1 \le i \le n$, then*
(1). On every V_j with $j \ne i$, we have $I_{g_i} = \pm 2$ is a constant.
(2). On V_i, I_{g_i} is not a constant, hence it is not a constant on X_0 either.

Proof. (1) follows from the definition of V_j and Proposition 2.1.

For (2), suppose I_{g_i} were a constant on V_i, then $I_{g_i} = I_{g_i}(\chi_0) = \pm 2$. Consider algebraic subset V of X_0 defined by the n equations $I^2_{\mu_j} = 4$, $(j \neq i)$, and $I^2_{g_i} = 4$. Then V_i is contained in some irreducible component V_0 of V containing χ_{ρ_0}. Hence $\dim V_0 \geq 1$. Contradiction to Proposition 2.1. □

For each $i = 1, \cdots, n$, let p_i be the projection map from $X(T_1) \times \cdots \times X(T_n)$ to the i-th factor $X(T_i)$. Denote by $r_i : X_0 \longrightarrow X(T_i)$ the composite of r and p_i. Then we have

Proposition 2.5. *For every $i = 1, \ldots, n$, the Zariski closure W_i of the image $r_i(V_i)$ in $X(T_i)$ has dimension* 1.

Proof. It is sufficient to consider the case $i = 1$. Since $\dim V_1 = 1$ and r_1 is regular, $\dim W_1 \leq 1$. Assume that $\dim W_1 = 0$. This means that $r_1(V_1)$ consists of a single point. Therefore, for any $g_1 \in \mathrm{Im}(\pi_1(T_1) \to \pi_1(M_L))$, I_{g_1} is a constant on V_1. This contradicts Lemma 2.4 part 2. □

2.3. For $1 \leq i \leq n$, denote by $R_D(T_i)$ the subvariety of $R(T_i)$ which consists of the diagonal representations. For such a representation ρ, by taking the eigenvalues of $\rho(\mu_i)$ and $\rho(\lambda_i)$, it is clear that $R_D(T_i)$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. We shall denote the coordinates by (l_i, m_i). Let $t_{i|D}$ be the restriction of t_i on $R_D(T_i) = \mathbb{C}^* \times \mathbb{C}^*$. Set $D_i = t^{-1}_{i|D}(W_i)$. By the proof of [LW1, Proposition 3.3], D_i is either irreducible or has two isomorphic irreducible components. Let $y^i \in D_i$ be the point corresponding to the character of the representation of the hyperbolic structure on $S^3 - L$. Let Y_i be an irreducible component of D_i containing y^i. Then Y_i is an algebraic curve. Denote by $\overline{Y_i}$ the smooth projective model of Y_i. Denote $\mathbb{C}(\overline{Y_i})$ the function field of $\overline{Y_i}$ which is isomorphic to the function field $\mathbb{C}(Y_i)$ of Y_i.

Definition 2.6. $Y^h = \overline{Y_1} \times \overline{Y_2} \times \cdots \times \overline{Y_n}$. *It is an n-dimensional smooth projective variety.*

Let $\mathbb{C}(Y^h)$ be the function field of Y^h. For each i, we have the injective morphism $j_i : \mathbb{C}(Y_i) = \mathbb{C}(\overline{Y_i}) \to \mathbb{C}(Y^h)$ which is induced by the i-th projection from Y^h to $\overline{Y_i}$. In this way we shall take the $\mathbb{C}(Y_i)$ as subfields of $\mathbb{C}(Y^h)$. This also induces the map j on the K-groups:

$$j : \bigoplus_{i=1}^{n} K_2(\mathbb{C}(Y_i)) \to K_2(\mathbb{C}(Y^h)).$$

For $f_i, g_i \in \mathbb{C}(Y_i)$, $i = 1, \cdots, n$, $j(\sum_{i=1}^n \{f_i, g_i\}) = \prod_{i=1}^n \{f_i, g_i\}$, where we identify f_i, g_i as rational functions on Y^h via the injection j_i. Note in this paper we shall use the multiplication in K_2 instead of addition.

Proposition 2.7. *There exists a finite field extension F of $\mathbb{C}(Y^h)$ with the property that for every $i = 1, \cdots, n$, there is a representation*

$$P_i : \pi_1(M_L) \longrightarrow SL_2(F)$$

such that for $1 \leq j \leq n$, if $j \neq i$, then traces of $P_i(\lambda_j)$ and $P_i(\mu_j)$ are either 2 or -2. If $j = i$, then

$$P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \text{ and } P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.$$

Proof. By definition, for each i, W_i is the Zariski closure of $r_i(V_i)$ in $X(T_i)$ and Y_i is mapped dominatingly to W_i. The canonical morphism $t : R_0 \to X_0$ is surjective, so we can choose a curve $D_i \subset R_0$ such that $t(D_i)$ is dense in V_i. Hence $r_i \circ t : D_i \to W_i$ is dominating. Then the function fields $\mathbb{C}(D_i)$ and $\mathbb{C}(Y_i)$ are finite extensions of $\mathbb{C}(W_i)$. By [CS1, Page 115], there is a tautological representation $p_i : \pi_1(M_L) \to SL_2(\mathbb{C}(D_i))$, and for any $g \in \pi_1(M_L)$ the trace of $p_i(g) = I_g$. Let F_i be the composite field of $\mathbb{C}(D_i)$ and $\mathbb{C}(Y_i)$. It is finite over both $\mathbb{C}(D_i)$ and $\mathbb{C}(Y_i)$. We shall view p_i as a representation in $SL_2(F_i)$. Since $t(D_i)$ is dense in V_i, by Lemma 2.4, if $j \neq i$, traces of $p_i(\lambda_j)$ and $p_i(\mu_j)$ are ± 2; if $j = i$, traces of $p_i(\lambda_i)$ and $p_i(\mu_i)$ are non-constant functions on D_i. Since $p_i(\lambda_i)$ and $p_i(\mu_i)$ are commuting and their eigenvalues l_i, m_i are in F_i, p_i is conjugate in $GL_2(F_i)$ to a representation

$$P_i : \pi_1(M_L) \longrightarrow SL_2(F_i)$$

such that if $j \neq i$, then traces of $P_i(\lambda_j)$ and $P_i(\mu_j)$ are either 2 or -2. If $j = i$, then

$$P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \text{ and } P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.$$

Fix an algebraic closure $\overline{\mathbb{C}(Y^h)}$ of $\mathbb{C}(Y^h)$. As above, by viewing $\mathbb{C}(Y_i)$ as a subfield of $\mathbb{C}(Y^h)$, we can identify the finite field extension F_i as a subfield of $\overline{\mathbb{C}(Y^h)}$. In $\overline{\mathbb{C}(Y^h)}$, take the composite of F_i and $\mathbb{C}(Y^h)$ over $\mathbb{C}(Y_i)$, denoted it by K_i. Then $F_i \subset K_i$ and K_i is a finite extension of $\mathbb{C}(Y^h)$ because the extension $F_i/\mathbb{C}(Y_i)$ is finite. Now let F be the composite of the fields $K_1, \cdots, K_n$ in $\overline{\mathbb{C}(Y^h)}$. Then F is a finite extension of $\mathbb{C}(Y^h)$ since each K_i is. Now compose each P_i with the embedding $SL_2(F_i) \hookrightarrow SL_2(F)$ and the proposition follows. □

3. K-theory and Deligne Cohomology

First we shall recall the basic definitions of K_2 of a commutative ring A. The reference is [Mil]. Let $GL(A)$ be the direct limit of the groups $GL_n(A)$, and let $E(A)$ be the direct limit of the groups $E_n(A)$ generated by all $n \times n$ elementary matrices.

Definition 3.1. *For $n \geq 3$, the Steinberg group $St(n, A)$ is the group defined by generators x_{ij}^{λ}, $1 \leq i \neq j \leq n$, $\lambda \in A$, subject to the following three relations:*

(i) $x_{ij}^{\lambda} \cdot x_{ij}^{\mu} = x_{ij}^{\lambda+\mu}$;
(ii) $[x_{ij}^{\lambda}, x_{jl}^{\mu}] = x_{il}^{\lambda\mu}$, *for* $i \neq l$;
(iii) $[x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1$, *for* $j \neq k$, $i \neq l$.

We have the canonical homomorphism $\phi_n : St(n, A) \to GL_n(A)$ by $\phi(x_{ij}^{\lambda}) = e_{ij}^{\lambda}$, where $e_{ij}^{\lambda} \in GL_n(A)$ is the elementary matrix with entry λ in the (i, j) place. Take the direct limit as $n \to \infty$, we get

$$\phi : St(A) \to GL(A).$$

Its image $\phi(St(A))$ is equal to $E(A)$, the commutator subgroup of $GL(A)$.

Definition 3.2. $K_2(A) = Ker\,\phi$.

It is well-known that $K_2(A)$ is the center of the Steinberg group $St(A)$ (See [Mil, Theorem 5.1]) and there is a canonical isomorphism $\alpha : H_2(E(A); \mathbb{Z}) \to K_2(A)$ (See [Mil, Theorem 5.10]).

3.1. **The Symbol.** Let U,V be two commutative elements of $E(A)$. Choose $u, v \in St(A)$ such that $U = \phi(u)$ and $V = \phi(v)$. Then the commutator $[u, v] = uvu^{-1}v^{-1}$ is in the kernel of ϕ. Hence $[u, v] \in K_2(A)$. We can check it is independent of the choices of u and v, and denote it by $U\bigstar V$.

Lemma 3.3. *(1). The construction is skew-symmetric:* $U\bigstar V = (V\bigstar U)^{-1}$.
(2). It is bi-multiplicative: $(U_1 \cdot U_2)\bigstar V = (U_1\bigstar V) \cdot (U_2\bigstar V)$.
(3). It is invariant under conjugation: if $P \in GL(A)$*, then* $(PUP^{-1})\bigstar(PVP^{-1}) = U\bigstar V$.

Proof. This is [Mil, Lemma 8.1]. For (3), we remark that since $E(A)$ is a normal subgroup of $GL(A)$, the left-hand side of the formula makes sense. If P,U,V are in $GL(n, A)$, then choose $p \in St(A)$ such that

$$\phi(p) = \begin{bmatrix} P & 0 \\ 0 & P^{-1} \end{bmatrix} \in E(A).$$

Now we have $\phi(pup^{-1}) = PUP^{-1}$ and $\phi(pvp^{-1}) = PVP^{-1}$. Hence,

$$[pup^{-1}, pvp^{-1}] = p[u, v]p^{-1} = [u, v].$$

□

Given two units f, g of A, consider the matrices:

$$D_f = \begin{bmatrix} f & 0 & 0 \\ 0 & f^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad D'_g = \begin{bmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{bmatrix}.$$

They are in $E(A)$ and commutative. Define the symbol $\{f, g\} := D_f\bigstar D'_g$.

Lemma 3.4. *(1). The symbol* $\{f, g\}$ *is skew-symmetric:* $\{f, g\} = \{g, f\}^{-1}$.
(2). It is bi-multiplicative: $\{f_1f_2, g\} = \{f_1, g\}\{f_2, g\}$.
(3). Denote $diag(f_1, \cdots, f_n)$ *the diagonal matrix with diagonal entries* $f_1, \cdots, f_n$*. If* $f_1 \cdots f_n = g_1 \cdots g_n = 1$*, then*

$$diag(f_1, \cdots, f_n)\bigstar diag(g_1 \cdots, g_n) = \{f_1, g_1\}\{f_2, g_2\} \cdots \{f_n, g_n\}.$$

where the right-hand side means the product of the symbols $\{f_i, g_i\}$, $1 \le i \le n$.

Proof. [Mil, Lemma 8.2 Lemma 8.3]. □

Let F be a field. Let $SL(F)$ be the direct limit of the groups $SL_n(F)$. We know that $SL(F) = E(F)$ and any element of $SL_n(F)$ is also naturally an element of $E(F)$.

Lemma 3.5. *Let* $u, t \in F$*, then*

(1).

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \bigstar \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = 1.$$

(2).

$$\begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} \bigstar \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \bigstar \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} \bigstar \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}$$

are 2-torsions in $K_2(F)$.
(3). If U *and* V *are two commuting matrices in* $SL_2(F)$ *and their traces are* 2 *or* -2*, then* $U\bigstar V$ *is a* 2*-torsion in* $K_2(F)$*. In particular if both have trace* 2*, then* $U\bigstar V = 1$.

Proof. We shall use the following notations. For $s \in F$,

$$M(1,s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \text{ and } M(-1,s) = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}.$$

In particular, $M(1,0)$ is the 2×2 identity matrix and $M(-1,0)$ is the 2×2 diagonal matrix with diagonal entries -1.

For (1), $M(1,t)\bigstar M(1,u) = [x_{12}^t, x_{12}^u] = 1$ by the definition of $St(A)$.

For (2), notice that by the definition, $M(1,0)\bigstar A = 1$ and $A\bigstar A = 1$ for any $A \in E(F)$. By Lemma 3.3,

$$1 = (M(-1,0)\cdot M(-1,0))\bigstar M(1,s) = (M(-1,0)\bigstar M(1,s))^2,$$

so $M(-1,0)\bigstar M(1,s)$ is a 2-torsion in $K_2(F)$. Since

$$M(-1,t) = M(-1,0)\cdot M(1,-t),\ M(-1,u) = M(-1,0)\cdot M(1,-u),$$

by Lemma 3.3 and the first part, we have

$$M(-1,t)\bigstar M(1,u) = (M(-1,0)\bigstar M(1,u))(M(1,-t)\bigstar M(1,u)) = M(-1,0)\bigstar M(1,u)$$

and

$$M(-1,t)\bigstar M(-1,u) = (M(-1,0)\bigstar M(1,-u))(M(1,-t)\bigstar M(-1,0)),$$

hence they are 2-torsion.

For (3), we can find $P \in GL_2(F)$ such that

$$PUP^{-1} = \begin{bmatrix} \pm 1 & t \\ 0 & \pm 1 \end{bmatrix} \text{ and } PVP^{-1} = \begin{bmatrix} \pm 1 & u \\ 0 & \pm 1 \end{bmatrix}.$$

Then it follows from the first two parts and Lemma 3.3 (3). □

The following proposition slightly generalizes [CCGLS, Lemma 4.1]. The proof is the same.

Proposition 3.6. *Let π be a free abelian group of rank two with $\{e_1, e_2\}$ its basis. Let $f : \pi \to E(A)$ be a group homomorphism defined by $f(e_1) = U$, $f(e_2) = V$. Then there is a generator t of $H_2(\pi;\mathbb{Z})$ such that $\alpha(f_*(t)) = U\bigstar V$, where $\alpha : H_2(E(A);\mathbb{Z}) \to K_2(A)$ is the canonical isomorphism and $f_* : H_2(\pi;\mathbb{Z}) \to H_2(E(A);\mathbb{Z})$ is the homomorphism induced by f.*

Proof. Since π is abelian, U and V are commutative. $U\bigstar V$ is well-defined. Let F be a free group on $\{e_1, e_2\}$. The homomorphism f gives rise to the following commutative diagram of short exact sequences of groups:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & {[F,F]} \arrow[r] \arrow[d, "f_2"] & F \arrow[r] \arrow[d, "f_1"] & \pi \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & K_2(A) \arrow[r] & St(A) \arrow[r, "\phi"] & E(A) \arrow[r] & 0
\end{tikzcd}$$

where $f_2([e_1, e_2]) = U\bigstar V$. Apply the homology spectral sequence to the above diagram, we obtain the following diagram:

$$\begin{tikzcd}
H_2(\pi;\mathbb{Z}) \arrow[r] \arrow[d, "f_*"] & H_0(\pi; H_1([F,F];\mathbb{Z})) \arrow[d, "g"] \\
H_2(E(A);\mathbb{Z}) \arrow[r, "\alpha"] & K_2(A)
\end{tikzcd}$$

The top horizontal arrow is an isomorphism. The class of $[e_1, e_2]$ is the generator of $H_0(\pi; H_1([F, F]; \mathbb{Z}))$. It is mapped to $U\bigstar V$ by g which is induced by f_2. Let t be the generator of $H_2(\pi; \mathbb{Z})$ mapped to the class of $[e_1, e_2]$. Then we have $\alpha(f_*(t)) = U\bigstar V$ by the commutative diagram. $\square$

Corollary 3.7. *(1). If $U = diag(u, u^{-1})$ and $V = diag(v, v^{-1})$, where u, v are units of A, then there is a generator t of $H_2(\pi; \mathbb{Z})$ such that $\alpha(f_*(t)) = \{u, v\}^2$.*
(2). Suppose A is a field. If U and V are two commuting matrices in $SL_2(A)$ and their traces are 2 or -2, then the image of any generator of $H_2(\pi; \mathbb{Z})$ is a 2-torsion in $K_2(A)$.

Proof. For (1), by Lemma 3.4, we have $U\bigstar V = \{u, v\}\{u^{-1}, v^{-1}\} = \{u, v\}^2$.
For (2), by Lemma 3.5 (3), $U\bigstar V$ is a 2-torsion in $K_2(F)$. $\square$

Now we can prove the following:

Theorem 3.8. *For each $i = 1, \cdots, n$, there is an integer $\epsilon(i) = 1$ or -1, such that the symbol $\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)}$ is a torsion element in $K_2(\mathbb{C}(Y^h))$.*

Proof. First, by Proposition 2.7, for each $i = 1, \cdots, n$, there exists a finite extension F of $\mathbb{C}(Y^h)$ and a representation $P_i : \pi_1(M_L) \to SL_2(F)$ such that for $1 \le j \le n$, if $j \ne i$, then traces of $P_i(\lambda_j)$ and $P_i(\mu_j)$ are either 2 or -2; if $j = i$, then

$$P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \text{ and } P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.$$

The inclusions of $\pi_1(T_i)$ into $\pi_1(M_L)$ induce the homomorphisms $\pi_1(T_i) \to E(F)$ by composing with P_i. This gives rise to the following homomorphisms on the group homology:

$$\bigoplus_{i=1}^n H_2(\pi_1(T_i), \mathbb{Z}) \xrightarrow{\alpha} H_2(\pi_1(M_L), \mathbb{Z}) \xrightarrow{\beta} H_2(E(F), \mathbb{Z}) = K_2(F), \tag{3.1}$$

where $\alpha = j_{1*} + \cdots + j_{n*}$, $\beta = P_{1*} + \cdots + P_{n*}$; j_{i*} are the morphisms on the group homology induced by the inclusions $j_i : \pi_1(T_i) \hookrightarrow \pi_1(M_L)$, and P_{i*} are those induced by P_i.

The orientation of M_L induces the orientation on each boundary torus T_i. Let $[T_i]$ be the oriented class of $H_2(T_i, \mathbb{Z}) = \mathbb{Z}$. By Corollary 3.7 (1), for each i, there is a generator ξ_i of $H_2(\pi_1(T_i))$ such that $P_{i*}(j_{i*}(\xi_i)) = \{l_i, m_i\}^2$. Since T_i is the $K(\pi_1(T_i), 1)$ space, $H_2(\pi_1(T_i), \mathbb{Z}) = H_2(T_i, \mathbb{Z})$. If $\xi_i = [T_i]$, define $\epsilon(i) = 1$; if $\xi_i = -[T_i]$, then define $\epsilon(i) = -1$.
Since L is a hyperbolic link, M_L is a $K(\pi_1(M_L), 1)$ space. Hence we have $H_2(\pi_1(M_L), \mathbb{Z}) = H_2(M_L, \mathbb{Z})$. Under this identification,

$$\alpha(\epsilon(1)\xi_1, \cdots, \epsilon(n)\xi_n) = \sum_{i=1}^n [T_i] = [\partial M_L] = 0 \text{ in } H_2(M_L, \mathbb{Z}).$$

Therefore,

$$\beta(\alpha(\epsilon(1)\xi_1, \cdots, \epsilon(n)\xi_n)) = 1 \text{ in } K_2(F). \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}\beta(\alpha(\epsilon(1)\xi_1,\cdots,\epsilon(n)\xi_n)) &= \beta(\sum_{i=1}^{n} j_{i*}(\epsilon(i)\xi_i))\\ &= \sum_{k=1}^{n} P_{k*}(\sum_{i=1}^{n} j_{i*}(\epsilon(i)\xi_i))\\ &= \sum_{i=1}^{n} P_{i*}(j_{i*}(\epsilon(i)\xi_i)) + \sum_{1\le i\neq k\le n} P_{k*}(j_{i*}(\epsilon(i)\xi_i))\\ &= \prod_{i=1}^{n}\{l_i,m_i\}^{2\epsilon(i)}\cdot\prod_{1\le i\neq k\le n} P_k(\mu_i)\bigstar P_k(\lambda_i),\end{aligned}$$

where the last step follows from Proposition 3.6 and Corollary 3.7. Note also we use multiplication in $K_2(F)$.

By Corollary 3.7 (2), $\prod_{1\le i\neq k\le n} P_k(\mu_i)\bigstar P_k(\lambda_i)$ is a 2-torsion. Compare with (3.2), we see that $\prod_{i=1}^{n}\{l_i,m_i\}^{2\epsilon(i)}$ is a 2-torsion in $K_2(F)$. By the same argument in [LW2, Proposition 3.2], $\prod_{i=1}^{n}\{l_i,m_i\}^{\epsilon(i)}$ is a torsion in $K_2(\mathbb{C}(Y^h))$. □

Remark 3.1. *This theorem is a natural generalization of our previous result* [LW2, Proposition 3.2] *about the hyperbolic knot case.*

3.2. **Deligne cohomology.** Let X be a nonsingular variety over $\mathbb{C}$. First let us recall the definition of the (holomorphic) Deligne cohomology groups of X. For more details, see [EV]. We define the complex $\mathbb{Z}(p)_{\mathscr{D}}$ of sheaves on X as follows:

$$\mathbb{Z}(p)_{\mathscr{D}}:\mathbb{Z}(p)\longrightarrow\mathcal{O}_X\xrightarrow{d}\Omega^1_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X, \tag{3.3}$$

where $\mathbb{Z}(p)$ is the constant sheaf $(2\pi i)^p\mathbb{Z}$ and sits in degree zero, $\mathcal{O}_X$ is the sheaf of holomorphic functions on X, and Ω^i_X is the sheaf of holomorphic i-forms on X. The first map in (3.3) is the inclusion and d is the exterior differential. The Deligne cohomology groups of X are defined as the hypercohomology of the complex $\mathbb{Z}(p)_{\mathscr{D}}$:

$$H^q_{\mathscr{D}}(X;\mathbb{Z}(p)) := \mathbb{H}^q(X;\mathbb{Z}(p)_{\mathscr{D}}).$$

For example, the exponential exact sequence of sheaves on X

$$0\to\mathbb{Z}(1)\to\mathcal{O}_X\to\mathcal{O}^*_X\to 0$$

gives rise to a quasi-isomorphism between $\mathbb{Z}(1)_{\mathscr{D}}$ and $\mathcal{O}^*_X[-1]$, where $\mathcal{O}^*_X$ is the sheaf of non-vanishing holomorphic functions on X. Moreover there is a quasi-isomorphism between $\mathbb{Z}(2)_{\mathscr{D}}$ and the complex ([EV, page 46])

$$\mathcal{O}^*_X\xrightarrow{d\log}\Omega^1_X[-1].$$

Therefore, we have for any integer q,

$$H^q_{\mathscr{D}}(X;\mathbb{Z}(1)) = H^{q-1}(X;\mathcal{O}^*_X);\quad H^q_{\mathscr{D}}(X;\mathbb{Z}(2)) = \mathbb{H}^{q-1}(X;\mathcal{O}^*_X\to\Omega^1_X).$$

On the other hand, Degline ([De]) interprets $\mathbb{H}^1(X;\mathcal{O}^*_X\to\Omega^1_X) = H^2_{\mathscr{D}}(X;\mathbb{Z}(2))$ as the group of holomorphic line bundles with (holomorphic) connections over X.

Let $\mathbb{C}(X)$ be the function field of X. Given two functions $f, g \in \mathbb{C}(X)$, let $D(f,g)$ be the divisors of the zeros and poles of f and g. $|D(f,g)|$ denotes its support. Then we have the morphism:

$$(f,g) : X - |D(f,g)| \longrightarrow \mathbb{C}^* \times \mathbb{C}^*,$$

given by $(f,g)(x) = (f(x), g(x))$.

Let $\mathcal{H}$ be the Heisenberg line bundle with connection on $\mathbb{C}^* \times \mathbb{C}^*$. For its construction, see [Bl, Ram]. Pull back $\mathcal{H}$ along (f,g), we obtain a line bundle with connection on $X - |D(f,g)|$, denoted by $r(f,g)$. Hence $r(f,g) \in \mathbb{H}^1(V; \mathcal{O}_V^* \to \Omega_V^1) = H^2_{\mathscr{D}}(V; \mathbb{Z}(2))$, where $V = X - |D(f,g)|$. Moreover we can represent $r(f,g)$ in terms of Čech cocycles for $\mathbb{H}^1(V; \mathcal{O}_V^* \to \Omega_V^1)$. Indeed, choose an open covering $(U_i)_{i\in I}$ of V such that the logarithm of f is well-defined on every U_i, denoted by $\log_i f$. Then $r(f,g)$ is represented by the cocyle (c_{ij}, ω_i), with

$$c_{ij} = g^{\frac{1}{2\pi i}(\log_j f - \log_i f)}, \quad \text{on } U_i \cap U_j; \tag{3.4}$$

$$\omega_i = \frac{1}{2\pi i} \log_i f \, \frac{dg}{g}, \quad \text{on } U_i. \tag{3.5}$$

Its curvature is

$$R = \frac{1}{2\pi i} \frac{df}{f} \wedge \frac{dg}{g}. \tag{3.6}$$

Remark 3.2. *There is a cup product $\cup$ on the Deligne cohomology groups (see* [Be, EV]*). For $f, g \in H^0(X; \mathcal{O}_X^*) = H^1_{\mathscr{D}}(X; \mathbb{Z}(1))$ as above, the cup product $f \cup g$ is exactly the line bundle $r(f,g) \in H^q_{\mathscr{D}}(X; \mathbb{Z}(2))$.*

Furthermore, we have the following properties about $r(f,g)$:

Lemma 3.9. *$r(f_1 f_2, g) = r(f_1, g) \otimes r(f_2, g)$, $r(f,g) = r(g,f)^{-1}$, and the Steinberg relation $r(f, 1-f) = 1$ holds if $f \neq 0$, $f \neq 1$.*

Proof. See [Bl, EV] and [Ram, Section 4]. The proofs there assume that X is a curve. But they are valid for arbitrary X without change. Note that in order to prove the Steinberg relation, we need the ubiquitous dilogarithm function. $\square$

Now proceed as in [Bl, Ram] for curves, we have the regulator map:

$$r : K_2(\mathbb{C}(X)) \longrightarrow \varinjlim_{U \subset X:\text{Zariski open}} H^2_{\mathscr{D}}(U; \mathbb{Z}(2))$$

Notice that when $\dim X > 1$, the only difference is that the line bundle $r(f,g)$ is not necessarily flat. However, we have the following

Proposition 3.10. *If $x \in K_2(\mathbb{C}(X))$ is a torsion, then the corresponding line bundle $r(x)$ is flat.*

Proof. Let U be the Zariski open subset over which the line bundle $r(x)$ is defined. Since x is a torsion in $K_2(\mathbb{C}(X))$, $r(x)$ is a torsion in $\mathbb{H}^1(U; \mathcal{O}_U^* \to \Omega_U^1)$. Choose a suitable open covering $(U_i)_{i\in I}$ of U such that $r(x)$ is represented by Čech cocycle (c_{ij}, ω_i) with $c_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ and $\omega_i \in \Omega^1(U_i)$. Then there exists an integer $n > 0$, such that the class represented by the cocycle $((c_{ij})^n, n\omega_i)$ is zero. Hence, there exists $t_i \in \mathcal{O}_X^*(U_i)$ (or by a refinement covering of $\{U_i\}$) , such that

$$c_{ij}^n = \frac{t_j}{t_i}, \quad \omega_i = \frac{1}{n} \frac{dt_i}{t_i}.$$

Therefore, $d\omega_i = 0$ for all i and the curvature is 0. □

Let $|D|$ be the support of the divisors of zeros and poles of the rational functions m_i, l_i on Y^h, $1 \le i \le n$. Define $Y_0^h = Y^h - |D|$. The line bundle $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ is well-defined over Y_0^h.

Corollary 3.11. *The line bundle $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ over Y_0^h is flat, therefore it is an element of $H^1(Y_0^h; \mathbb{C}^*)$.*

Proof. This follows from Theorem 3.8 and Proposition 3.10. □

By the Čech cocycle for $r(f, g)$ given in (3.4) and (3.5), we can represent $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ as follows. Choose an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of Y_0^h such that on every U_α, the logarithms of l_i are well-defined and denoted by $\log_\alpha l_i$. Then $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ is represented by the cocyle $(c_{\alpha\beta}, \omega_\alpha)$:

$$c_{\alpha\beta} = \prod_{i=1}^n l_i^{\epsilon(i)[\frac{1}{2\pi i}(\log_\beta l_i - \log_\alpha l_i)]}, \quad \text{on } U_\alpha \cap U_\beta; \tag{3.7}$$

$$\omega_\alpha = \sum_{i=1}^n \frac{\epsilon(i)}{2\pi i}(\log_\alpha l_i)\,\frac{dm_i}{m_i}, \quad \text{on } U_\alpha. \tag{3.8}$$

Let $t_0 = (l_1^0, m_1^0, \cdots, l_n^0, m_n^0) \in Y_0^h$ be a point corresponding to the hyperbolic structure of the link complement $S^3 - L$. Then the monodromy of the flat line bundle $r(\prod_{i=1}^n \{m_i, l_i\}^{\epsilon(i)})$ give rises to the representation $M : \pi_1(Y_0^h, t_0) \to \mathbb{C}^*$. With its explicit descriptions (3.7), (3.8), we have the following formula for M. Let γ be a loop based at t_0. Let $\log l_i$ be a branch of logarithm of l_i over $\gamma - \{t_0\}$, then we have (c.f.[De, (2.7.2)])

$$M(\gamma) = \exp\{\sum_{i=1}^n (-\frac{\epsilon(i)}{2\pi i})(\int_\gamma \log l_i\,\frac{dm_i}{m_i} - \log m_i(t_0)\int_\gamma \frac{dl_i}{l_i})\}. \tag{3.9}$$

Theorem 3.12. *(i) The real 1-form $\eta = \sum_{i=1}^n \epsilon(i)(\log|l_i|\,d\arg m_i - \log|m_i|\,d\arg l_i)$ is exact on Y_0^h. Hence there exists a smooth function $V : Y_0^h \to \mathbb{R}$ such that*

$$dV = \sum_{i=1}^n \epsilon(i)(\log|l_i|\,d\arg m_i - \log|m_i|\,d\arg l_i).$$

(ii) Suppose $m_i^0 = 2$, $1 \le i \le n$. For a loop γ with initial point t_0 in Y_0^h

$$\frac{1}{4\pi^2}\sum_{i=1}^n \epsilon(i)\int_\gamma (\log|m_i|\,d\log|l_i| + \arg l_i\,d\arg m_i) = \frac{p}{q},$$

where p is some integer and q is the order of the symbol $\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)}$ in $K_2(\mathbb{C}(Y^h))$.

Proof. First, by (3.8), the curvature of the flat line bundle is

$$R = \sum_{i=1}^n \frac{\epsilon(i)}{2\pi i}(\frac{dl_i}{l_i} \wedge \frac{dm_i}{m_i}) = 0.$$

On the other hand, we have $d\eta = \mathrm{Im}(\sum_{i=1}^n \epsilon(i)(\frac{dl_i}{l_i} \wedge \frac{dm_i}{m_i}))$, hence η is a real closed 1-form.

Since the symbol $\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)}$ has order q in $K_2(\mathbb{C}(Y^h))$, for a loop γ in Y_0^h, by (3.9) we have

$$1 = M(\gamma)^q = (\exp\{\sum_{i=1}^n (-\frac{\epsilon(i)}{2\pi i})(\int_\gamma \log l_i \frac{dm_i}{m_i} - \log m_i(t_0) \int_\gamma \frac{dl_i}{l_i})\})^q.$$

Write $\sum_{i=1}^n \epsilon(i)(\int_\gamma \log l_i \frac{dm_i}{m_i} - \log m_i(t_0) \int_\gamma \frac{dl_i}{l_i}) = Re + iIm$, where Re and Im are the real and imaginary parts respectively. Then we have $\exp(\frac{q \cdot Im}{2\pi} + \frac{q \cdot Re}{2\pi i}) = 1$. Therefore, $Im = 0$ and $\frac{q \cdot Re}{2\pi i} = 2\pi i p$, for some integer p.

A straightforward calculation or [LW2, Lemma 3.4] shows that

$$Im = \int_\gamma \eta, \quad Re = -\sum_{i=1}^n \epsilon(i) \int_\gamma (\log|m_i|\, d\log|l_i| + \arg l_i\, d\arg m_i) = \int_\gamma \xi. \tag{3.10}$$

These immediately imply both parts of the theorem. □

Remark 3.3. *When $n = 1$, our V is (up to sign) the volume function of the representation of the knot complement (*[Dun]*). For $n \geq 2$, up to some constant and signs related to the orientations on each boundary component of the hyperbolic link complement, the function V should be closely related to the volume function given in* [Ho, Theorem 5.5].

Remark 3.4. *If there exists any representation $\rho : \pi_1(Y^h) \to GL_n(\mathbb{C}), n \geq 2$, then Reznikov* [Re, Theorem 1.1] *proved that the Chern classes $c_i \in H^{2i}_{\mathcal{D}}(Y^h, \mathbb{Z}(i))$ in the Deligne cohomology groups are torsion for all $i \geq 2$.*

3.3. **On the Bohr-Sommerfield quantization condition for hyperbolic links.** In this section, we shall discuss the above Theorem 3.12(ii) from symplectic point of view. When $n = 1$, this is the Bohr-Sommerfield quantization condition proposed by Gukov for knots [Guk, Page 597], and is proved in [LW2, Theorem 3.3 (2)].

Let Σ be a closed surface with fundamental group π. Its $SL_2(\mathbb{C})$-character variety is the space of equivalence classes of representations from π into $SL_2(\mathbb{C})$. This variety carries a natural complex-sympelctic structure, where a complex-symplectic structure is a nondegenerate closed holomorphic exterior 2-form (see [Go1, Go2]).

A homomorphism $\rho : \pi \to SL_2(\mathbb{C})$ is irreducible if it has no proper linear invariant subspace of $\mathbb{C}^2$, and irreducible representations are stable points, denoted by $\mathrm{Hom}(\pi, SL_2(\mathbb{C}))^s$. Now $SL_2(\mathbb{C})$ acts freely and properly on $\mathrm{Hom}(\pi, SL_2(\mathbb{C}))^s$, and the quotient $X^s(\Sigma) = \mathrm{Hom}(\pi, SL_2(\mathbb{C}))^s/SL_2(\mathbb{C})$ is an embedding onto an open subset in the geometric quotient $\mathrm{Hom}(\pi, SL_2(\mathbb{C}))//SL_2(\mathbb{C})$. Thus $X^s(\Sigma)$ is a smooth irreducible complex quasi-affine variety which is dense in the geometric quotient (see [Go2, Section 1]). Note that ρ is a nonsingular point if and only if $\dim Z(\rho)/Z(SL_2(\mathbb{C})) = 0$, and this corresponds to the top stratum $X^s(\Sigma)$, where $Z(u)$ is the centralizer of u in $SL_2(\mathbb{C})$. If $\rho \in \mathrm{Hom}(\pi, SL_2(\mathbb{C}))$ is a singular point (i.e., $\dim Z(\rho)/Z(SL_2(\mathbb{C})) > 0$), then all points of $\sigma \in \mathrm{Hom}(\pi, Z(Z(\rho)))^s$ with $\mathrm{stab}(\sigma) = Z(\sigma) = Z(\rho)$ have the same orbit type and form a stratification of the $SL_2(\mathbb{C})$-character variety (see [Go1, Section 1]).

We have the $SL_2(\mathbb{C})$-character variety $X(T^2)$ of the torus T^2 as a surface in $\mathbb{C}^3$ given by

$$x^2 + y^2 + z^2 - xyz - 4 = 0.$$

See [LW1, Proposition 3.2]. There exists a natural symplectic structure on the smooth top stratum $X^s(T^2)$ of $X(T^2)$, and there exists a symplectic structure ω on the character variety $X^s(\partial M_L) = \prod_{i=1}^n X^s(T_i^2)$ such that $X(M_L) \cap X^s(\partial M_L)(\subset X(M_L))$ is a Lagrangian subvariety of $X^s(\partial M_L)$, where $X^s(\partial M_L)$ is a smooth irreducible variety which is open and dense in $X(\partial M_L)$.

The inclusion $\partial M_L \to M_L$ indeed induces a degree-one map on the irreducible components. Thus $r(X_0)^s$ (the smooth part of the image $r(X_0)$) is a Lagrangian submanifold of the symplectic manifold $X^s(\partial M_L)$. Note that the pullback of the symplectic 2-form on the double covering of $X^s(T_i^2)$ is again a skew-symmetric and nondegenerate. The symplectic form $\tilde{\omega}_i$ through the map t_i on the irreducible component gives the Lagrangian property for the corresponding pullback of the Lagrangian part $r(X_0^i)^s$. Hence we have the product Lagrangian smooth part of the pullback of $\prod_{i=1}^n r(X_0^i)^s$. Then we need to see that the smooth projective model preserves the Lagrangian and symplectic property.

Let $\tilde{X}(T_i^2)$ be the symplectic blowup of the double covering of $X(T_i^2)$ as in [MS]. The blowup in the complex category carries a natural symplectic structure on $\tilde{X}(T_i^2)$ ([MS, Section 7.1]). On the other hand, the corresponding part $\overline{Y}_i$ of Y_i (the irreducible component of D_i containing y_i) lies in the symplectic manifold $\tilde{X}(T_i^2)$.

Define a compatible Lagrangian blowup with respect to the complex blowup as following. Define a real submanifold $\tilde{\mathbb{R}}^n$ of $\mathbb{R}^n \times \mathbb{R}P^{n-1}(\subset \mathbb{C}^n \times \mathbb{C}P^{n-1})$ as a subspace of pairs (x, l) with $x = Re(z) \in l$, where $l \in \mathbb{R}P^{n-1}$ is a real line in $\mathbb{R}^n$. If $I_{\mathbb{C}}$ is an complex conjugation on $\mathbb{C}^n$ and $J_{\mathbb{C}P^{n-1}}$ be the complex involution on $\mathbb{C}P^{n-1}$ as complex conjugation on each components, then

$$\tilde{\mathbb{R}}^n = Fix(I_{\mathbb{C}} \times J_{\mathbb{C}P^{n-1}}|_{\tilde{\mathbb{C}}^n}) \subset \tilde{\mathbb{C}}^n = \{(z_1, \cdots, z_n; [w_1 : \cdots : w_n]) | w_j z_k = w_k z_j, 1 \le j, k \le n\}.$$

It is clear that $\tilde{\mathbb{R}}^n$ is Lagrangian in $\tilde{\mathbb{C}}^n$. Hence the real Lagrangian blowup $\tilde{\overline{Y}}_i$ is Lagrangian in $\tilde{X}(T_i^2)$, and the Lagrangian submanifold $\tilde{Y}^h$ is Lagrangian in the symplectic manifold $\prod_{i=1}^n \tilde{X}(T_i^2)$. This only gives a way to have the symplectic and Lagrangian properties being preserved under the blowup, and treat the Lagrangian blowup in a real blowup with respect to the complex one.

Now we have a Lagrangian submanifold $\tilde{Y_0^h}$ in a symplectic manifold. Suppose $m_i^0 = 2$, $1 \le i \le n$. For a loop γ with initial point t_0 in $\tilde{Y_0^h}$, by Theorem 3.12(ii)

$$\frac{1}{4\pi^2} \sum_{i=1}^n \epsilon(i) \int_\gamma (\log|m_i| \, d\log|l_i| + \arg l_i \, d\arg m_i) = \frac{p}{q},$$

where p is some integer and q is the order of the symbol $\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)}$ in $K_2(\mathbb{C}(Y^h))$. We will call this result the Bohr-Sommerfield quantization condition for hyperbolic links. It would be interesting to give an interpretation from mathematical physics, as what Gukov did for hyperbolic knots.

4. On a possible unified Volume Conjecture for both knots and links

In this section, we shall give some descriptions and speculations of a possible parametrized volume conjecture which includes both hyperbolic knots and links.

By Corollary 3.11, the class $r(\prod_{i=1}^n \{m_i, l_i)^{\varepsilon_i})$ corresponds to a flat line bundle over Y_0^h, therefore the curvature of the holomorphic connection is zero. Formally this can be expressed

as $d(\xi + i\eta) = 0$, where ξ and η are defined in (3.10). Hence, $\frac{1}{2\pi i}(\xi + i\eta)$ can be viewed as the 1-*form* Chern-Simons of the line bundle $r(\prod_{i=1}^n \{m_i, l_i)^{\varepsilon_i})$.

Let $\gamma : [0,1] \to Y_0^h$ be a path with initial point $\gamma(0) = t_0$ the point corresponding to the complete hyperbolic structure. Write $\gamma(t) = (m(t), l(t)) = (m_1(t), l_1(t), \cdots, m_n(t), l_n(t))$. Recall that q is the order of the symbol $\prod_{i=1}^n \{m_i, l_i\}^{\varepsilon_i}$ in $K_2(\mathbb{C}(Y^h))$. Let $Vol(L)$ and $CS(L)$ be the volume and usual Chern-Simons invariant of the complete hyperbolic structure on $S^3 - L$ respectively. Now according to Theorem 3.12, we define

$$V(\gamma(1)) = Vol(L) + 2 \cdot \sum_{i=1}^n \epsilon(i) \int_\gamma (\log |l_i| \, d\arg m_i - \log |m_i| \, d\arg l_i). \tag{4.1}$$

$$U(\gamma(1)) = 4\pi^2 CS(L) + q \cdot \sum_{i=1}^n \epsilon(i) \int_\gamma (\log |m_i| \, d\log |l_i| + \arg l_i \, d\arg m_i). \tag{4.2}$$

and call (4.2) the *special Chern-Simons invariant* of the hyperbolic link L at $\gamma(1)$. By Theorem 3.12, the quantity $\frac{\sqrt{-1}}{2\pi}(V(\gamma(1)) + \frac{\sqrt{-1}}{2\pi}U(\gamma(1)))$ lies in $\mathbb{C}/\mathbb{Z}$.

Remark 4.1. *The above $U(\gamma(1))$ in (4.2) is different from the usual Chern-Simons invariant for a 3-dimensional manifold. The latter comes from the transgressive 3-form of the second Chern class of the 3-dimensional manifold.*

When $\gamma(1)$ varies in a neighborhood of t_0, the $(\mathbb{C}^*)^n$-parametrized invariant $\frac{\sqrt{-1}}{2\pi}(V(\gamma(1)) + \frac{\sqrt{-1}}{2\pi}U(\gamma(1)))$ is the generalization of the right-hand side of [LW2, Conjecture 3.9].

In order to formulate a conjecture parallel to the knot case as in [LW2, Conjecture 3.9], we have to find a way of relating the quantum invariants to the n-dimensional variety Y_0^h which comes from the $SL_2(\mathbb{C})$ character variety. By the work of Kashaev and Baseilhac-Benedetti ([BB, Ka1]), there exists an $SL_2(\mathbb{C})$ quantum hyperbolic invariant for a hyperbolic link in S^3, which is conjectured to give the information of the volume and Chern-Simons at the point for the complete hyerbolic struture. We speculate that there should exist a "family" version of their quantum hyperbolic invariants parametrized by Y_0^h, then we can replace the left-hand side of [LW2, Conjecture 3.9] by its logarithm limit to formulate the generalized volume conjecture for a hyperbolic link in S^3.

Here is a conjectural description. First we assume that for a point $p \in Y_0^h$ near t_0 of the complete hyperbolic structure, we can define certain quantum invariants $K_N(L, p)$. Then for fixed number a_j, $1 \leq j \leq n$, we take $m_j = -\exp(i\pi a_j)$, $1 \leq j \leq n$, in (4.1) and (4.2). We formulate the following:

Conjecture: (*A Possibly Unified Paremetrized Volume Conjecture*)

$$\lim_{N\to\infty} \frac{\log K_N(L, \gamma(1))}{N} = \frac{1}{2\pi}(Vol(\gamma(1)) + i\frac{1}{2\pi}U(\gamma(1))). \tag{4.3}$$

Remark 4.2. *When L is a hyperbolic knot (i.e., $n = 1$), we can take $K_N(L, \gamma(1))$ as the colored Jones polynomial $J_N(K, e^{2\pi i a/N})$. In this cae our unified Conjecture 4.3 is reduced to* [LW2, The Reformulated Generalized Volume Conjecture (3.9)] *for hyperbolic knots.*

Remark 4.3. *When $n \geq 2$, we can take $K_N(L, \gamma(0)) = K_N(L, t_0)$ as the the Kashaev and Baseilhac-Benedetti invariant which is well-defined and conjectured to give the information of the volume and Chern-Simons at the complete hyperbolic structure t_0. For general p, although we expect that there is a way of deforming $K_N(L, t_0)$ to get $K_N(L, p)$, we do not have a rigorous definition.*

Remark 4.4. *Since the usual colored Jones polynomial formulation of volume conjecture does not hold for all links (*[MMOTY]*), it is a very interesting question to see what is really behind the volume conjecture for links. From the regulator point of view developed here, we expect that our unifed volume conjecture (4.3) is a good candidate which gives a $(\mathbb{C}^*)^n$-paramatrized version of the volume conjecture for both links and knots.*

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